Transverse Sound Waves

Reinhold Brückner
(*1963, Konstanz)

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1 - Amplitude model for acoustic waves

Look first at a 1-d gas. See figure 1.

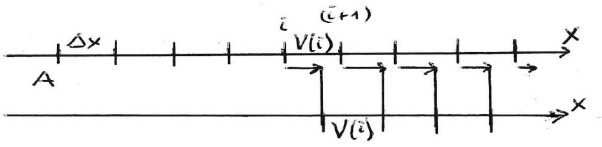

figure 1

Subdivide gas in small volumes V(i) of width Δx bounded by some "membrane" at i to the left and i+1 to the right. (See figure 1 top row with membranes at equilibrium position)
Displacement of these membranes is described by an amplitude A(i) (figure 1 middle row)
and the displaced volume V(i) (figure 1 bottom row) becomes

$$(1.1) \qquad V(i) = \Delta x + \Delta A = \left(1 + \frac{\Delta A}{\Delta x}\right) V_0$$

with $\Delta A = A(i+1) - A(i)$ and $V_0 = \Delta x$ the unperturbed volume

From (1.1) it is clear that we need

$$(1.2) \qquad \frac{\Delta A}{\Delta x} > -1$$

amplitudes must not "overtake" each other.

The molar density n changes from the unperturbed density n_0 accordingly

$$(1.3) \qquad n(i) = \frac{n_0}{\left(1 + \frac{\Delta A}{\Delta x}\right)}$$

Let us switch from discrete variables (i) to continuous ones (x) and to differentiation.
For small amplitude variations, we expand to first order

$$(1.4) \qquad n(x) = \frac{n_0}{\left(1 + \frac{\partial A}{\partial x}\right)} \approx \left(1 - \frac{\partial A}{\partial x}\right) n_0$$

We are interested in the density variation around the unperturbed value

$$(1.5) \qquad \Delta n(x) = -\left(\frac{\partial A}{\partial x}\right) n_0$$

It seems useful to limit amplitude gradient, which is dimensionless to

$$(1.6) \qquad \left|\frac{\partial A}{\partial x}\right| < 1$$

Does this restriction still allow a useful theory?
For an acoustic frequency of 1m wavelength, one has an amplitude limit of 10 cm.
Not even the most powerful loudspeakers in the world can achieve that.

Generalisation to three dimensions gives for the density variation

$$(1.7) \qquad \Delta n(x) = -\langle \nabla | \vec{A} \rangle n_0$$

where the divergence replaces the 1-d derivative.

2 - Equation of motion

The density change gives rise to a pressure change

$$(2.1) \qquad \Delta p = m \frac{dP}{d(mn)} \Delta n = mc^2 \Delta n = -n_0 mc^2 \langle \nabla | A \rangle$$

m is the molar mass, c is the velocity of sound.
The negative pressure gradient is the force per volume, related to volume acceleration by Newton's law. Volume coordinates are given by the amplitude **A**. (vectors in text are denoted **bold**)

$$(2.2) \qquad n_0 mc^2 \frac{\partial^2 \vec{A}(r,t)}{c^2 \partial t^2} = -\nabla \Delta p(r,t) = n_0 mc^2 \nabla \langle \nabla | A(r,t) \rangle$$

$$(2.3) \qquad n_0 mc^2 \left(\frac{\partial^2 \vec{A}(r,t)}{c^2 \partial t^2} - \nabla \langle \nabla | A(r,t) \rangle\right) = 0$$

This is the wave equation for free longitudinal sound waves.

The Lagrange density for the free sound field is

$$(2.4) \qquad L = n_0 mc^2 \left(\frac{1}{2} \langle \frac{\partial A(r,t)}{c \partial t} | \frac{\partial A(r,t)}{c \partial t} \rangle - \frac{1}{2} \langle \nabla | A(r,t) \rangle \langle \nabla | A(r,t) \rangle\right)$$

Its energy density is simply

$$(2.5) \qquad H = n_0 mc^2 \left(\frac{1}{2} \langle \frac{\partial A(r,t)}{c \partial t} | \frac{\partial A(r,t)}{c \partial t} \rangle + \frac{1}{2} \langle \nabla | A(r,t) \rangle \langle \nabla | A(r,t) \rangle\right)$$

The energy flow vector (acoustic Poynting vector; power per surface) is

(2.6) $$\vec{P} = -n_0 mc^2 c\left(\frac{\partial \vec{A}(r,t)}{c\,\partial t}\langle \nabla | A(r,t)\rangle\right)$$

There can be additional forces (per volume) which contribute a term on the right-hand side (RHS) of the form **f**(**r**+**A**,t).
Why **r**+**A** and not **r** ? Because the volume in question is located at (**r** +**A**(**r**))
If we assume smoothly varying force fields and small amplitudes, we can expand the force field to first order

(2.7) $$\vec{f}(\vec{r}+\vec{A}) \approx \vec{f}(\vec{r}) + \nabla_r \langle f(\vec{r}) | A\rangle$$

Assuming the force gradient small, as well as the amplitude, we feel entitled to set

(2.8) $$\vec{f}(\vec{r}+\vec{A}) \approx \vec{f}(\vec{r})$$

The total, inhomogeneous, wave equation then becomes

(2.9) $$n_0 mc^2\left(\frac{\partial^2 \vec{A}(r,t)}{c^2\,\partial t^2} - \nabla\langle \nabla | A(r,t)\rangle\right) = \vec{f}(r,t)$$

3 - Green functions

The Green function (GF) (understood as a distribution) relates the force field on the RHS to the resultant amplitude field.
In all what follows, we integrate the factor n_0mc^2 into **f** on RHS and set the time variable (t) to (ct)

(3.1) $$\vec{A}(r,t) = \int d^4x' [G(r,t,r',t')]\vec{f}(r',t')$$

The GF of vector fields is a matrix . In the simplest case, it is the unit matrix times a scalar function.

Pressure field

As derived above (2.1), pressure is related to (div **A**) whose dynamic equation is

(3.2) $$\left(\frac{\partial^2 \langle \nabla | \vec{A}(r,t)\rangle}{\partial t^2} - \nabla^2\langle \nabla | A(r,t)\rangle\right) = \langle \nabla | f(r,t)\rangle$$

The pressure GF is obtained as

(3.3) $$G_P = \frac{1}{R}\delta(T-R)$$

where $T = t - t'$ (t > t') and $R = |\vec{r} - \vec{r}\,'|$

Amplitude field

Recall the wave equation (new units)

(3.4) $$(\frac{\partial^2 \vec{A}(r,t)}{\partial t^2} - \nabla \langle \nabla | A(r,t) \rangle) = \vec{f}(r,t)$$

The differential operator acts differently on rotational ("transverse") or irrotational ("longitudinal") amplitude fields.
Transverse fields are defined by

(3.5) $$\langle \nabla | A_T \rangle = 0$$

Amplitude and force fields can be linearly decomposed into their longitudinal and transverse parts. The equation becomes

(3.6) $$(\frac{\partial^2 \vec{A}_L(r,t)}{\partial t^2} - \nabla \langle \nabla | A_L(r,t) \rangle) + (\frac{\partial^2 \vec{A}_T(r,t)}{\partial t^2}) = \vec{f}_L(r,t) + \vec{f}_T(r,t)$$

The equation for either part (L/T) must be fulfilled simultaneously, hence

(3.7) $$(\frac{\partial^2 \vec{A}_L(r,t)}{\partial t^2} - \nabla \langle \nabla | A_L(r,t) \rangle) = \vec{f}_L(r,t)$$

(3.8) $$(\frac{\partial^2 \vec{A}_T(r,t)}{\partial t^2}) = \vec{f}_T(r,t)$$

We therefore obtain two different GF for longitudinal and transverse fields, respectively.

Transverse Green function

That is easy. The amplitude is just the double time integral of the RHS, with a spatial deltafunction which takes care of the spatial integral, see (3.1)

(3.9) $$G_T = \iint^t dt' \, \delta(\vec{r} - \vec{r}')$$

Longitudinal Green function

Any longitudinal function can be expressed as the gradient of a scalar funtion (C).
The longitudinal equation becomes

(3.10) $$\nabla (\frac{\partial^2 C_A(r,t)}{\partial t^2} - \nabla^2 C_A(r,t)) = \nabla C_f(r,t)$$

For the scalar function one gets the same GF as for the pressure

(3.11) $$G_L = \frac{1}{R} \delta(T - R)$$ we recall $T = t - t'$ (t > t') and $R = |\vec{r} - \vec{r}'|$

Longitudinal and Transverse Parts

To obtain transverse and longitudinal parts for any vector field **C**(**r**,t), one uses

(3.12) $$\nabla^2 \frac{1}{|\vec{r}-\vec{r}\,'|} = -4\pi\,\delta^3(\vec{r}-\vec{r}\,')$$

and

(3.13) $$\nabla^2 \vec{C} = \nabla\langle\nabla|\vec{C}\rangle - \nabla\, X\,(\nabla\, X\,\vec{C})$$

to get

(3.14)
$$\vec{C}(\vec{r}) = -\nabla^2 \int d^3x'\, \frac{\vec{C}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} = -\nabla \int d^3x'\, \frac{\langle\vec{C}(\vec{r}\,')|\nabla\rangle}{4\pi|\vec{r}-\vec{r}\,'|} + \nabla\, X \int d^3x'\, \frac{\nabla\, X\,\vec{C}(\vec{r}\,')}{4\pi|\vec{r}-\vec{r}\,'|} = \vec{C}_L + \vec{C}_{tr}$$

Acoustic Larmor Oscillator

Suppose an homogeneous force field in a small volume V_C oscillating in time

$$\vec{f}(\vec{r},t) = \vec{f}\sin(\omega t) \quad \text{for} \quad \vec{r} \in V_C \;\; ; \;\; \vec{f}(\vec{r},t) = 0 \quad \text{otherwise}$$

This field is the linear superposition of equal strength of an electric dipole and a magnetic dipole where we interpret respective electric and magnetic fields as amplitudes. See figure 2.

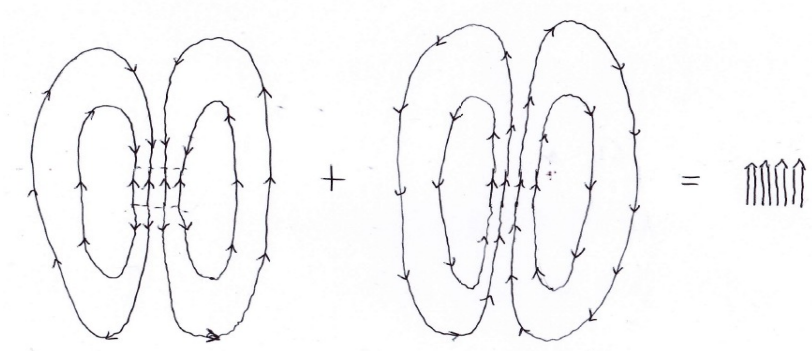

figure 2: left is electric dipole; middle magnetic dipole, right is resultant localised field

Electric and magnetic potentials are given by (up to signs and factors)
(3.15)
$$\Phi_E(\vec{r}) = \int d^3x'\, \frac{\langle\vec{f}/2|(\vec{r}-\vec{r}\,')\rangle}{|\vec{r}-\vec{r}\,'|^3} \;\rightarrow\; \vec{f}_L(\vec{r}) = \nabla \int d^3x'\, \frac{\langle\vec{f}/2|(\vec{r}-\vec{r}\,')\rangle}{|\vec{r}-\vec{r}\,'|^3}$$
$$\vec{\Phi}_M(\vec{r}) = \int d^3x'\, \frac{(\vec{f}/2)\,X\,(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} \;\rightarrow\; \vec{f}_{tr}(\vec{r}) = \nabla\, X \int d^3x'\, \frac{(\vec{f}/2)\,X\,(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}$$

Outside of V_C , longitudinal and transverse fields exactly cancel each other.
Inside V_C they add up, each contributing a field **f** /2 .
From the force field, we get the amplitude field by applying longitudinal and transverse Green functions to the respective parts of the force field

(3.16) $\vec{A}(\vec{r},t)=\int dt'\int d^3V'_C(G_L+G_{tr})\vec{f}_L(\vec{r}',t')+\int dt'\int d^3\bar{V}'_C(G_L-G_{tr})\vec{f}_L(\vec{r}',t')$

We have divided the integration volume in V_C and its complement $\bar{V}_C$ and used that force fields add up / subtract in these volumes.
Since $G_L-G_{tr}\neq 0$ we get a contribution from all of space to our amplitude field.
A localised loudspeaker acts as if it sounded everywhere in space.
The dropoff is not even fast (power law), since we have logarithmic hearing.

Do we observe this ? No, we don't.
Sure, there are purely longitudinal sources (acoustic monopoles) for whom the transverse field vanishes and which give a localised emission. But even that is an idealisation. An ordinary loudspeaker is a mixture of acoustic monopole, dipole and higher orders and will always have an extended longitudinal and transverse field, thus creating the uncomfortable situation encountered.

How can we remedy this situation?

There is only one way: We have to complement the differential wave equation to recover the scalar differential operator $\partial_t^2-\nabla^2$ that is, we have to postulate T*ransverse Acoustic Waves (TAWs)*.

The wave equation then becomes

(3.17) $n_0mc^2(\frac{\partial^2\vec{A}(r,t)}{c^2\partial t^2}+\nabla X(\nabla X\vec{A}(r,t))-\nabla\langle\nabla|A(r,t)\rangle)=\vec{f}(r,t)$

The free-field Lagrange density becomes
(3.18)
$$L=n_0mc^2(\frac{1}{2}\langle\frac{\partial A(r,t)}{c\partial t}|\frac{\partial A(r,t)}{c\partial t}\rangle-\frac{1}{2}(\nabla X\vec{A}(r,t))(\nabla X\vec{A}(r,t))-\frac{1}{2}\langle\nabla|A(r,t)\rangle\langle\nabla|A(r,t)\rangle)$$

The free-field energy density is
(3.19)
$$H=n_0mc^2(\frac{1}{2}\langle\frac{\partial A(r,t)}{c\partial t}|\frac{\partial A(r,t)}{c\partial t}\rangle+\frac{1}{2}(\nabla X\vec{A}(r,t))(\nabla X\vec{A}(r,t))+\frac{1}{2}\langle\nabla|A(r,t)\rangle\langle\nabla|A(r,t)\rangle)$$

The acoustic Poynting vector now has two parts, longitudinal and transverse

(3.20) $\vec{P}=-n_0mc^2c[\frac{\partial\vec{A}(r,t)}{c\partial t}\langle\nabla|A(r,t)\rangle+\frac{\partial\vec{A}(r,t)}{c\partial t}X(\nabla X\vec{A}(r,t))]$

For physical reasons, we have requested ‖div**A**‖ < 1. We might do the same thing with ‖curl **A**‖ < 1, which has a simple physical meaning, see figure 3.

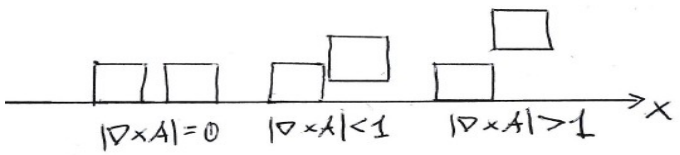

figure 3: two adjacent volume elements, with transverse relative displacements

TAW Properties

The Poynting vector also gives momentum flow. At equal power, TAWs produce the same momentum as longitudinal acoustic waves (LAWs). A microphone will then not distinguish between the two types. Selection, rather TAW elimination, will occur when one has waveguides leading to the microphone. LAWs are rather insensitive to boundaries since motion parallel to a boundary and hence in the direction of wave propagation is generally allowed.
This is not the case with TAWs, which behave like electrodynamic radiation (they can also be polarized) and get deformed and damped in narrow waveguides.
This ioccurs in the human ear which sports such a waveguide with the eardrum at its end.
Transverse sound is *silent sound*.

All wind instruments being basically pipes, will only emit LAWs.
String instruments are capable of emitting either of them but loudspeakers will not reproduce the original, recording microphones being insensitive to polarisation.

Conclusion

The necessity of TAWs is a mathematical one at first and one might question its physical viability. We are accustomed to pressure forces but attribute shear gas forces to friction, denying gas internal cohesion. Yet, transposing pressure forces from macroscosm to microcosm supposes cohesion in some small "volume elements" which can react to pressure as a whole and do not just mix.
TAWs just tell us that, microscopically, this cohesion also applies to shear displacements.
Let us underline that the model presented is viable only for *small oscillations* where the notion of an equilibrium position makes sense. It would be wrong and against observation to apply this model to aerodynamic flow.